THE SEMILINEAR COUPLED SYSTEMS FOR THE EXTERNAL DAMPING MODELS WITH VARIABLE COEFFICIENTS

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ABSTRACT. We present in this article some results on the sufficient conditions for the global solvability with the arbitrarily small data of the Cauchy problem for the following semilinear coupled system with variable coefficient

\[
\begin{align*}
    u_{tt} + a(x)(-\Delta)\sigma u + u_t &= F(D|\alpha v, v_t), \\
    v_{tt} + a(x)(-\Delta)\sigma v + v_t &= G(D|\alpha u, u_t) \\
    u(0, x) &= u_0(x), \quad u_t(0, x) = u_1(x), \\
    v(0, x) &= v_0(x), \quad v_t(0, x) = v_1(x).
\end{align*}
\]

The nonlinearities of our interest are $(F, G) = (||D|\alpha v|p, ||D|\alpha u|q)$, or $(F, G) = (|v|p, |u|q)$, where the parameter $\sigma$ satisfies $\sigma \in (0, 1)$. We will show that the “critical exponents” $p, q$ for the small data global solvability have a close relation to the established exponent of the corresponding semilinear problems for the external damping equations.

Keywords: external damping, coupled systems, global (in time) solvability, decay estimates, small data solutions.

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1. INTRODUCTION

In [6] Nishihara and Wakasugi studied the Cauchy problem of the weakly coupled system of the damped wave equation

\[
\begin{align*}
    u_{tt} - \Delta u + u_t &= |v|^p, \quad t \geq 0, x \in \mathbb{R}^n, \\
    v_{tt} - \Delta v + v_t &= |u|^q, \quad t \geq 0, x \in \mathbb{R}^n, \\
    (u, u_t, v, v_t)(0, x) &= (u_0, u_1, v_0, v_1)(x), \quad x \in \mathbb{R}^n
\end{align*}
\]

(1.1)

They proved that global existence of small data solutions to (1.1) holds, in any space dimension $n \geq 1$ provided that

\[
\alpha := \max\{p, q\} + 1 < \frac{n}{2}.
\]

(1.2)

It can be checked directly that condition (1.2) is equivalent to

\[
\max\{p, q\}(\min\{p, q\} + 1 - p_F(n)) > p_F(n)
\]

(1.3)

where $p_F(n) = 1 + 2/n$ is the Fujita critical exponent for the damped wave equation.

The aim of our study is to obtain the analogous results in the case of fractional damping models using the decay estimates for the linear external damping models. Denote $p_E = p_E(\sigma, \alpha) := \frac{n + 2\sigma}{n + \alpha}$. We will show that for the Cauchy problem for
system
\[
\begin{aligned}
&u_{tt} + a(-\Delta)^{\alpha}u + u_t = |D|^{\alpha}v|^p, \quad t \geq 0, x \in \mathbb{R}^n, \\
v_{tt} + a(-\Delta)^{\alpha}v + v_t = |D|^{\alpha}u|^q, \quad t \geq 0, x \in \mathbb{R}^n,
\end{aligned}
\]
(1.4)
\[(u, u_t, v, v_t)(0, x) = (u_0, u_1, v_0, v_1)(x), \quad x \in \mathbb{R}^n,
\]
where \(0 < a = \text{const}, |D| := (-\Delta)^{\frac{1}{2}}\), the global existence of the small data solutions holds if
\[
\max\{p, q\}(\min\{p, q\} + 1 - p_E) > p_E
\]
for all \(n \geq 2\).

For the Cauchy problem with nonlinearities \((F, G) = (|v_t|^p, |u_t|^q)\) we will show that
\[
\begin{aligned}
&u_{tt} + a(x)(-\Delta)^{\alpha}u + u_t = |v_t|^p, \quad t \geq 0, x \in \mathbb{R}^n, \\
v_{tt} + a(x)(-\Delta)^{\alpha}v + v_t = |u_t|^q, \quad t \geq 0, x \in \mathbb{R}^n,
\end{aligned}
\]
(1.6)
\[(u, u_t, v, v_t)(0, x) = (u_0, u_1, v_0, v_1)(x), \quad x \in \mathbb{R}^n
\]
a solvability result in the class \(H^n\) for sufficient large \(s\) will be guaranteed by condition
\[
p, q > p_S
\]
where the exponent \(p_S\) has a relation with the global solvability of the following Cauchy problem
\[
u_{tt} + a(x)(-\Delta)^{\alpha}u + u_t = |u_t|^p, \quad u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x).
\]
(1.8)

We recall that the general semi-linear structurally damping model is a family of following problems containing the fractional Laplacians with parameters \(\delta\) and \(\sigma\)
\[
u_{tt} + (-\Delta)^{\alpha}u + (-\Delta)^{\delta}u_t = F(u, u_t, |D|^{\alpha}u),
\]
(1.9)
\[u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x),
\]
(see \([1, 2, 3, 7]\) for the considered damping models).

In general, without any restriction on the parameter \(\sigma\), we can define the fractional Laplacian by the Fourier transform
\[
\mathcal{F}((-\Delta)^{\sigma} f(\xi)) = |\xi|^{2\sigma} \mathcal{F}(f)(\xi)
\]
for all \(\sigma > 0\), where \(\mathcal{F}(f)\) denotes the Fourier transform of the function \(f\) with respect to \(x\) variable.

For \(\sigma \in (0, 1)\) we can use the integral representation of the fractional Laplacian:
\[
(-\Delta)^{\sigma} u(x) = c_{n, \sigma} \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x - y|^{n+2\sigma}} dy
\]
for sufficient smooth \(u\), with a normalization positive constant \(c_{n, \sigma} = \frac{2^{2\sigma} \Gamma(n/2+\sigma)}{\pi^{n/2} \Gamma(1-\sigma)}\) depending on \(n\) and \(\sigma\).

For the external damping model, when \(\delta = 0\), using the diffusion phenomenon and the Markov property of the semigroup generated by non-negative self-adjoint operators, we have obtained in \([8, 9, 10]\) the following decay estimates for the solution of the linear external damping problem.
Proposition 1.1. Let \( a(x) \) be a continuous function satisfying
\[
(1.10) \quad a_1 \leq a(x) \leq a_2, \text{ for all } x \in \mathbb{R}^n,
\]
with positive constant \( a_1, a_2 \). Then the solution \( v(t, x) \) of the linear Cauchy problem for external damping model
\[
(1.11) \quad v_{tt} + a(x)(-\Delta)^s v + v_t = 0, \quad t \geq 0, x \in \mathbb{R}^n
\]
\[
\quad v(0, x) = v_0(x), \quad x \in \mathbb{R}^n
\]
\[
\quad v_t(0, x) = v_1(x), \quad x \in \mathbb{R}^n
\]
and its derivatives satisfy the following \((L^1 \cap L^2) - L^2\) estimates
\[
(1.12) \quad \|v(t, \cdot)\|_{L^2} \lesssim (1 + t)^{-\frac{n}{4s}} \|v_0\|_{L^1 \cap L^2} + (1 + t)^{-\frac{n}{4s}} \|v_1\|_{L^1 \cap H^{-s}},
\]
\[
(1.13) \quad \|v(t, \cdot)\|_{H^s} \lesssim (1 + t)^{-\frac{n}{4s} - \frac{1}{2}} \|v_0\|_{L^1 \cap H^s} + (1 + t)^{-\frac{n}{4s} - \frac{1}{2}} \|v_1\|_{L^1 \cap L^2},
\]
\[
(1.14) \quad \|v_t(t, \cdot)\|_{L^2} \lesssim (1 + t)^{-\frac{n}{4s} - 1} \|v_0\|_{L^1 \cap H^s} + (1 + t)^{-\frac{n}{4s} - 1} \|v_1\|_{L^1 \cap L^2},
\]
\[
(1.15) \quad \|v(t, \cdot)\|_{H^k} \lesssim (1 + t)^{-\frac{n}{4s} - \frac{k}{2s}} \|v_0\|_{L^1 \cap H^{k+s}} + (1 + t)^{-\frac{n}{4s} - \frac{k}{2s}} \|v_1\|_{L^1 \cap H^k}.
\]
In the case \( a = \text{const} > 0 \) we have also the following additional \(L^2 - L^2\) estimates
\[
\|v(t, \cdot)\|_{L^2} \lesssim \|v_0\|_{L^2} + (1 + t)\|v_1\|_{L^2},
\]
\[
\|v_t(t, \cdot)\|_{L^2} \lesssim (1 + t)^{-1}\|v_0\|_{H^s} + \|v_1\|_{L^2},
\]
\[
\|D^{\sigma}v(t, \cdot)\|_{L^2} \lesssim (1 + t)^{-\frac{1}{2}}\|v_0\|_{H^s} + (1 + t)^{-\frac{1}{2}}\|v_1\|_{L^2}.
\]

The above linear estimates are essential in the study of global solvability for various nonlinear Cauchy problems to damping models with arbitrary small initial data.

2. Main results

In this section we will state our main results.

First we consider the case where the nonlinearities have the form \( |u_t|^p \) and \( |v_t|^q \) with \( p, q > 2 \). We introduce the exponent
\[
p_s := s + 1 - \sigma
\]
for \( s > \sigma + \frac{n}{2} \).

Theorem 2.1. Let us consider the Cauchy problem for the following system of the nonlinear external damped model
\[
(2.1) \quad \begin{cases}
    u_{tt} + a(x)(-\Delta)^{\sigma} u + u_t = |v_t|^p, & t \geq 0, x \in \mathbb{R}^n, \\
    v_{tt} + a(x)(-\Delta)^{\sigma} v + v_t = |u_t|^q, & t \geq 0, x \in \mathbb{R}^n,
\end{cases}
\]
\[
(u, u_t, v, v_t)(0, x) = (u_0, u_1, v_0, v_1)(x), \quad x \in \mathbb{R}^n
\]
with \( a \) satisfies \( \sigma \in (0, 1) \) and \( n > 4\alpha \). The coefficient \( a(x) \) is supposed to be a continuous function satisfying condition (1.10). The data \((u_0, u_1, v_0, v_1)\) are assumed
to belong to the function space \( \left( (L^1 \cap H^s) \times (L^1 \cap H^{s-\sigma}) \right)^2 \) with \( s > \sigma + \frac{n}{2} \). Then for any
\[ p, q > p_S \]
there exists a uniquely determined global (in time) small data energy solution from \( \left( C([0, \infty), H^s) \cap C^1([0, \infty), H^{s-\sigma}) \right)^2 \).

Next, let’s consider the case \( (F, G) = (|D|^\alpha v|p, |D|^\alpha u|^q) \), where \( \alpha \in [0, \sigma) \) and \( 0 < a = \text{const.} \) For \( p_E = \frac{n + 2\sigma}{n + \alpha} \), put
\[
\gamma(p) := \frac{(n + \alpha)(p_E - p)_+}{2\sigma},
\]
\[
\gamma(q) := \frac{(n + \alpha)(p_E - q)_+}{2\sigma}
\]
- the parameters that represent the possible loss of decay with respect to the corresponding linear estimates for \( u \) and \( v \). We use here the notation \((k)_+ := \max\{k, 0\} \).

Note that \( p, q \) can be strictly smaller than \( p_E \) in the following consideration, that means \( \gamma(p) \) or \( \gamma(q) \) can be positive. Then we have the following global solvability result for problem (1.4).

**Theorem 2.2.** Let \( p, q \geq p_E - 1 \) and \( p, q \in \left[ 2, \frac{n}{n + 2(\alpha - \sigma)} \right) \) such that the conditions (1.5) holds for \( \sigma \in (0, 1) \). Moreover, suppose that the data \((u_0, u_1, v_0, v_1)\) are chosen from the space \( A := ((L^1 \cap H^s) \times (L^1 \cap L^2))^2 \). Then there exists \( \varepsilon > 0 \) such that for all \( n \geq 2 \) for any small data \((u_0, u_1, v_0, v_1)\) with
\[
A := \| (u_0, v_0) \|_{L^1 \cap H^s} + \| (u_1, v_1) \|_{L^1 \cap L^2} < \varepsilon
\]
there exists the (global) solution
\[
(u(t, x), v(t, x)) \in \left( C([0, \infty), H^s) \cap C^1([0, \infty), L^2) \right)^2
\]
to the Cauchy problem (1.4) with \( \alpha \in [0, \sigma) \). Moreover, the following estimates are satisfied
\[
\| u(t, \cdot) \|_{L^2} \lesssim (1 + t)^{-\frac{n}{4\sigma} - 1 + \gamma(p)} A,
\]
\[
\| v(t, \cdot) \|_{L^2} \lesssim (1 + t)^{-\frac{n}{4\sigma} + \gamma(q)} A,
\]
\[
\| u(t, \cdot) \|_{H^s} \lesssim (1 + t)^{-\frac{n}{4\sigma} - \frac{1}{2} + \gamma(p)} A,
\]
\[
\| v(t, \cdot) \|_{H^s} \lesssim (1 + t)^{-\frac{n}{4\sigma} - \frac{1}{2} + \gamma(q)} A,
\]
\[
\| u(t, \cdot) \|_{L^2} \lesssim (1 + t)^{-\frac{n}{4\sigma} - 1 + \gamma(p)} A,
\]
\[
\| v(t, \cdot) \|_{L^2} \lesssim (1 + t)^{-\frac{n}{4\sigma} - 1 + \gamma(q)} A,
\]
in the case \( p, q \neq p_E \). If \( p = p_E \) or \( q = q_E \) then the corresponding loss of decay \( (1 + t)^{\gamma(p)} \) or, respectively, \( (1 + t)^{\gamma(q)} \) is replaced by \( \log(e + t) \).
3. The proof of the main results

Proof. (of Theorem 2.1.)

Let’s denote
\[ A := \| (u_0, v_0) \|_{L^1 \cap H^s} + \| (u_1, v_1) \|_{L^1 \cap H^{s - \sigma}}. \]

We introduce the data space \( A := \left( (L^2 \cap H^s) \times (L^1 \cap H^{s - \sigma}) \right)^2 \) and the solution space \( X(t) = \left( C([0, t], H^s) \cap C^1([0, t], H^{s - \sigma}) \right)^2 \) with the norm
\[ \| (u, v) \|_{X(t)} = \sup_{0 \leq \tau \leq t} \left( W(u) + W(v) \right), \]
where
\[ W(u) = \left( f_0(\tau)^{-1} \| u(\tau, \cdot) \|_{L^2} + f_s(\tau)^{-1} \| D^s u(\tau, \cdot) \|_{L^2} + g(\tau)^{-1} \| u_t(\tau, \cdot) \|_{L^2} + g_{s - \sigma}(\tau)^{-1} \| D^{s - \sigma} u_t(\tau, \cdot) \|_{L^2} \right), \]
and similarly for \( v \).

From the estimates of Proposition 1.1 we can choose
\[ f_0(\tau) := (1 + \tau)^{-\frac{\sigma}{4}}, \quad f_s(\tau) := (1 + \tau)^{-\frac{\sigma}{4} - \frac{s}{n}}, \quad g(\tau) := (1 + \tau)^{-\frac{\sigma}{8} - 1}, \quad g_{s - \sigma}(\tau) := (1 + \tau)^{-\frac{\sigma}{8} - \frac{s}{n}}. \]

We define the integral operator \( N : (u, v) \in X(t) \rightarrow N[u, v] \in X(t) \) by:
\[ N[u, v] = L + (Fv, Gu), \]
where
\[ L = E_0(t, x) *_{x} (u_0, v_0)(x) + E_1(t, x) *_{x} (u_1, v_1)(x), \]
\[ F(v) = \int_0^t E_1(t - \tau, x) *_{x} | v(\tau, x) |^p d\tau, \]
\[ G(u) = \int_0^t E_1(t - \tau, x) *_{x} | u_t(\tau, x) |^q d\tau. \]

The local and global existence of small data solutions to (2.1) will follow by the standard contraction argument if we can show that for the exponents \( p, q \) satisfying the given conditions the estimates
\[ \| N[u, v] \|_{X(t)} \lesssim A + \| (u, v) \|_{X(t)}^p + \| (u, v) \|_{X(t)}^q, \]
and the Lipschitz property
\[ \| N[u, v] - N[\tilde{u}, \tilde{v}] \|_{X(t)} \lesssim \| (u, v) - (\tilde{u}, \tilde{v}) \|_{X(t)} \left( \| (u, v) \|_{X(t)}^{p - 1} + \| (\tilde{u}, \tilde{v}) \|_{X(t)}^{p - 1} + \| (u, v) \|_{X(t)}^{q - 1} + \| (\tilde{u}, \tilde{v}) \|_{X(t)}^{q - 1} \right) \]
must hold.

By the linear estimates (1.12)-(1.14) in Prop. 1.1, it immediately follows that
\[ \| L \|_{X(t)} \lesssim A. \]
Next we will estimate the $L^2$ norm of $Fv$ itself. To do that we apply the $(L^1 \cap L^2) - L^2$ estimates on the interval $[0, t]$ to conclude

\[
\|Fv(t, \cdot)\|_{L^2} \lesssim \int_0^t (1 + t - \tau)^{-\frac{n}{4\sigma}} \|u_4(\tau, \cdot)\|_{L^1 \cap L^2} d\tau. \tag{3.6}
\]

We get immediately from the definitions of $L_p$ norms that

\[
\|u_4(\tau, \cdot)\|_{L^1 \cap L^2} \lesssim \|u_4(\tau, \cdot)\|_{L^p}^p + \|u_4(\tau, \cdot)\|_{L^2}^2.
\]

To estimate the norm $\|u_4(\tau, \cdot)\|_{L^k}^k, k = 1, 2$, we apply the fractional Gagliardo-Nirenberg inequality from Proposition 4.1 in the following form

\[
\|w(\tau, \cdot)\|_{L^q} \lesssim \||D|^{s-\sigma} w(\tau, \cdot)\|_{L^2_{0,s-\sigma(q:2)}}^\theta \|w(\tau, \cdot)\|_{L^2}^{1-\theta}, \tag{3.7}
\]

with $w(\tau, \cdot) = u_4(\tau, \cdot)$, where for $q \geq 2$ we need

\[
\theta_{0,s-\sigma}(q, 2) = \frac{n}{s - \sigma} \left( \frac{1}{2} - \frac{1}{q} \right) \in [0, 1)
\]

that is, $2 < q$ if $\frac{n}{2(s - \sigma)} < 1$.

Since $\theta_{0,s-\sigma}(p, 2) < \theta_{0,s-\sigma}(2p, 2)$ we obtain on the interval $(0, t)$ the estimate

\[
\int_0^t (1 + t - \tau)^{-\frac{n}{4\sigma}} \|u_4(\tau, \cdot)\|_{L^1 \cap L^2} d\tau \lesssim (u, v)_{X(t)}^p \int_0^t (1 + t - \tau)^{-\frac{n}{4\sigma} - p \left( \frac{n}{4\sigma} + 1 + \frac{n}{2\sigma} \left( \frac{1}{2} - \frac{1}{p} \right) \right)} d\tau. \tag{3.8}
\]

Similarly we get the $L^2$ estimate for $Gu$.

Now we recall that for $\max\{\alpha; \beta\} > 1$ the inequality

\[
\int_0^t (1 + t - \tau)^{-\alpha}(1 + \tau)^{-\beta} d\tau \lesssim (1 + t)^{-\min(\alpha, \beta)} \tag{3.9}
\]

holds.

If $p, q > p_S$, then in this case,

\[
(p \left( \frac{n}{4\sigma} + 1 + \frac{n}{2\sigma} \left( \frac{1}{2} - \frac{1}{p} \right) \right) > \frac{n}{4\sigma} > 1,
\]

thus

\[
\int_0^t (1 + t - \tau)^{-\frac{n}{4\sigma} - p \left( \frac{n}{4\sigma} + 1 + \frac{n}{2\sigma} \left( \frac{1}{2} - \frac{1}{p} \right) \right)} d\tau \lesssim (1 + t)^{-\frac{n}{4\sigma}}. \tag{3.10}
\]

Therefore the $L^2$-norm of $Fv$ is bounded by $C(1 + t)^{-\frac{n}{4\sigma}} ((u, v)_{X(t)})^p$. Similarly for $Gu$.

By this approach we have proved the $L^2$ norm estimate in (3.4) for $Fv$ and $Gu$.

Differentiating the equations for $N[u, v]$ with respect to $t$ we obtain

\[
\partial_t N[u, v] = L_4 + \int_0^t \partial_t (G_1(t + \tau, x) *_{(x)} [Fv, Gu]) d\tau.
\]

Using the above techniques for getting the estimate for $Fv$ we arrive at

\[
(1 + \tau)^{\frac{n}{4\sigma} + 1} \|\partial_t Fv(\tau, \cdot)\|_{L^2} \leq C \|N[u, v]\|_{X(t)}^p \tag{3.11}
\]

for all $\tau \in [0, t]$ under the same assumption for $p, q$. 

Now let us turn to estimate \( \| \partial_k |D|^{s-\sigma} N[u(t, \cdot), v(t, \cdot)] \|_{L^2} \). We use the following

\[
\partial_k |D|^{s-\sigma} N[u, v] = \{D|^{s-1} L_k(t, x) + \int_0^t \partial_k |D|^{s-\sigma}(E_k(t-\tau, x)\phi(\tau))(|v_k(\tau, \cdot)|^p, |u_k(\tau, \cdot)|^p) d\tau. 
\]

Taking account of the linear estimate (1.16) with \( k = s - \sigma \) and using the \( (L^1 \cap L^2) - L^2 \) estimates on the interval \( (0, t) \) we obtain

\[
(3.12)
\]

\[
\| \partial_k |D|^{s-\sigma} (Fv) \|_{L^2} \lesssim \int_0^t (1 + t - \tau)^{-\frac{n+2(s+\sigma)}{4\sigma}} (\|v_k(\tau, \cdot)\|_{L^1 \cap L^2} + \|v_k(\tau, \cdot)\|_{\dot{H}^{s+\sigma}}) d\tau.
\]

The integral with \( \|v_k(\tau, \cdot)\|_{L^1 \cap L^2} \) will be estimated by the Gagliardo-Nirenberg inequality as before in the following manner

\[
\int_0^t (1 + t - \tau)^{-\frac{n+2(s+\sigma)}{4\sigma}} \|v_k(\tau, \cdot)\|_{L^1 \cap L^2} d\tau \lesssim \|(u, v)\|_{X(t)} \int_0^t (1 + t - \tau)^{-\frac{n+2(s+\sigma)}{4\sigma}} (1 + \tau)^{-\frac{n}{4\sigma} + \frac{n}{2\sigma} \left( \frac{n}{2} - \frac{1}{2} \right)} d\tau.
\]

In order to apply inequality (3.9) with \( a := \frac{n+2(s+\sigma)}{4\sigma} \) and \( b := p(\frac{n}{4\sigma} + 1 + \frac{n}{2\sigma} (\frac{n}{2} - \frac{1}{2})) \) we need the condition \( a \leq b \), that is equivalent to the following condition for \( p \):

\[
(3.13) \quad p > 1 + \frac{n+2(s+\sigma)}{2(n+2\sigma)}.
\]

To estimate the integrals with \( \|v_k(\tau, \cdot)\|_{\dot{H}^{s+\sigma}} \) we apply the composition result (see Corollary 4.6 in Appendix) for \( p > s \). Thus we can proceed further as follows:

\[
\int_0^t (1 + t - \tau)^{-\frac{n}{4\sigma} - \frac{s+\sigma}{2\sigma}} ||v_k(\tau, \cdot)||_{\dot{H}^{s+\sigma}} d\tau \lesssim \int_0^t (1 + t - \tau)^{-\frac{n}{4\sigma} - \frac{s+\sigma}{2\sigma}} ||v_k(\tau, \cdot)||_{\dot{H}^{s+\sigma}} ||v_k(\tau, \cdot)||_{\dot{H}^{s+\sigma}}^{-1} d\tau
\]

\[
\lesssim \int_0^t (1 + t - \tau)^{-\frac{n}{4\sigma} - \frac{s+\sigma}{2\sigma}} ||v_k(\tau, \cdot)||_{\dot{H}^{s+\sigma}} ||v_k(\tau, \cdot)||_{\dot{H}^{s+\sigma}}^{-1} d\tau.
\]

\[
(3.14)
\]

\[
\lesssim \int_0^t (1 + t - \tau)^{-\frac{n}{4\sigma} - \frac{s+\sigma}{2\sigma}} ||v_k(\tau, \cdot)||_{\dot{H}^{s+\sigma}} \left( ||v_k(\tau, \cdot)||_{L^2} + ||v_k(\tau, \cdot)||_{\dot{H}^{s+\sigma}} \right)^{p-1} d\tau,
\]

with \( s - \sigma > s_0 > \frac{n}{2} \).

Using again the linear estimates we get

\[
\int_0^t (1 + t - \tau)^{-\frac{n}{4\sigma} - \frac{s+\sigma}{2\sigma}} ||v_k(\tau, \cdot)||_{\dot{H}^{s+\sigma}} d\tau \lesssim 
\]

\[
(3.15) \quad \|(u, v)\|_{X(t)} \int_0^t (1 + t - \tau)^{-\frac{n}{4\sigma} - \frac{s+\sigma}{2\sigma}} (1 + \tau)^{-\frac{n}{4\sigma} - 1} 
\]

\[
\left( (1 + \tau)^{-\frac{n}{4\sigma} - 1} + (1 + \tau)^{-\frac{n}{4\sigma} - \frac{s+\sigma}{2\sigma}} \right)^{p-1} d\tau.
\]

It is obvious that for \( p > 2 \) and \( s > 1 \) the integral in (3.15) can be estimated as previously, thus it implies

\[
(1 + \tau)^{\frac{n}{4\sigma} + \frac{s+\sigma}{2\sigma}} ||v_k(\tau, \cdot)||_{\dot{H}^{s+\sigma}} \lesssim \|(u, v)\|_{X(t)}^p \quad \text{for all} \quad \tau \in (0, t).
\]
An analogous reasoning leads to the other estimates in (3.4) which are required in the definition of \(X(t)\)-norm.

The second inequality (3.5) is obtained by using Hölder’s and Kato - Ponce’s inequalities. Namely, the \(L^1 \cap L^2\)-norm of \(F(v_t(\tau, x)) - F(\tilde{v}_t(\tau, x))\) for \(F(v_t) = |v_t|^p\) is estimated by using

\[
|F(v_t) - F(\tilde{v}_t)| \lesssim |v_t - \tilde{v}_t|(|v_t|^{p-1} + |	ilde{v}_t|^{p-1}).
\]

Applying Hölder’s inequality we obtain

\[
\|F(v_t(\tau, \cdot)) - F(\tilde{v}_t(\tau, \cdot))\|_{L^1} \lesssim \|v_t(\tau, \cdot) - \tilde{v}_t(\tau, \cdot)\|_{L^p} \big(\|v_t(\tau, \cdot)\|_{L^p}^{p-1} + \|\tilde{v}_t(\tau, \cdot)\|_{L^p}^{p-1}\),
\]

\[
\|F(v_t(\tau, \cdot)) - F(\tilde{v}_t(\tau, \cdot))\|_{L^2} \lesssim \|v_t(\tau, \cdot) - \tilde{v}_t(\tau, \cdot)\|_{L^{2p}} \big(\|v_t(\tau, \cdot)\|_{L^{2p}}^{p-1} + \|\tilde{v}_t(\tau, \cdot)\|_{L^{2p}}^{p-1}\).
\]

The \(L^p\) and \(L^{2p}\) norms of the difference \(v_t - \tilde{v}_t\) are estimated again by the fractional Gagliardo-Nirenberg inequality. Therefore, they can be bounded from above by the norms of \(|v_t - \tilde{v}_t|_{L^2}\) and \(|v_t - \tilde{v}_t|_{H^{s-\sigma}}\) that are included in the norm of the space \(X(t)\).

To obtain other Sobolev norms estimates for the difference \((u, v) - (\tilde{u}, \tilde{v})\) we will use a version of Kato-Ponce inequality (see Proposition 4.7) which is formulated in terms of homogeneous Sobolev spaces. This inequality is also known as fractional Leibniz rule. Denote \(\gamma = s - \sigma\). We can estimate

\[
|v_t(s, \cdot)|^p - |\tilde{v}_t(s, \cdot)|^p \lesssim \int_0^1 \|v_t(s, \cdot) - \tilde{v}_t(s, \cdot)\|_{H^\gamma} \, d\theta + \|f(\theta v_t + (1-\theta)\tilde{v}_t(s, \cdot))\|_{H^\gamma},
\]

with \(f(w) = w|w|^{p-2}\).

Now we apply the Kato-Ponce inequality from Proposition 4.7 with the following constants \(p_1, q_1, p_2, q_2 > 0\):

\[
r = p_2 = q_1 = 2, p_1 = q_2 = \infty,
\]

to estimate the \(H^\gamma\)-norm of the product \(fg\), with \(f := f(\theta v_t + (1-\theta)\tilde{v}_t)\) and \(g := v_t - \tilde{v}_t\), in the right-hand side of (3.16) as follows.

\[
||D|^{\gamma} (fg)||_{L^2} \lesssim \|f\|_{L^\infty} ||D|^{\gamma} g||_{L^2} + ||D|^{\gamma} f||_{L^2} ||g||_{L^\infty}.
\]

With the assumption \(p - 1 > \gamma = s - \sigma\) the norm in \(H^{\gamma, 2} = H^\gamma\) of \(f(\theta v_t + (1-\theta)\tilde{v}_t)\) can be estimated if we apply a general composition result for homogeneous Sobolev spaces (see Corollary 4.6).

By this way we obtain the second inequality (3.5) for the operator \(N[u, v]\), since \(p - 1 > \gamma = s - \sigma\) is equivalent to our condition \(p > s + 1 - \sigma\) and both of the norms \(|v_t(t, \cdot)|_{L^2}\) and \(|v_t(t, \cdot)|_{H^\gamma}\) are included in the norm \(\|(u, v)|_{X(t)}\|\).

We summarize all conditions for the exponents \(p\) and \(q\) that have been derived from the above considerations:

- From the application of the Gagliardo - Nirenberg inequality: \(p, q > 2\)
- From (3.13): \(p, q > 1 + \frac{n\sigma + 2(s-\sigma)}{2(s+2\sigma)}\)
- From the above application of composition result: \(p, q > s + 1 - \sigma\).

It’s clear that these conditions are reduced to the last requirement

\[
p, q > s + 1 - \sigma,
\]
since \( s + 1 - \sigma > 2 \) and \( s + 1 - \sigma > 1 + \frac{n\sigma + 2(s - \sigma)}{2(n + 2\sigma)} \) for the given range \( s > \sigma + \frac{n}{2} \).

The required sufficient condition \( p, q > s + 1 - \sigma \) for the existence of global (in time) small data solutions to the Cauchy problem for the semilinear coupled system (2.1) in the whole space has been proved.

Theorem 2.1 is proved now completely. \( \square \)

**Proof.** (of Theorem 2.2) We introduce for all \( t > 0 \) the function spaces

\[
X(t) := \left( C([0, t], H^{s}) \cap C^{1}([0, t], L^{2}) \right)^{2}
\]

with the norm:

\[
\| (u, v) \|_{X(t)} := \sup_{0 \leq \tau \leq t} \left( (1 + \tau)^{-\gamma(p)} W(u) + (1 + t)\gamma(q) W(v) \right),
\]

where

\[
W(u) = (f_{0}(\tau)^{-1} \| u(\tau, \cdot) \|_{L^{2}} + f_{\sigma}(\tau)^{-1} \| D^{\sigma} u(\tau, \cdot) \|_{L^{2}} + g(\tau)^{-1} \| u(\tau, \cdot) \|_{L^{2}}),
\]

and similarly for \( v \). From the estimates of Proposition 1.1 we can choose

\[
f_{0}(\tau) := (1 + \tau)^{-\frac{2}{\sigma}}, \quad f_{\sigma}(\tau) := (1 + \tau)^{-\frac{2}{\sigma} - 1/2}, \quad g(\tau) := (1 + \tau)^{-\frac{1}{\sigma} - 1}.
\]

In the case \( p = p_{E} \) (or \( q = p_{E} \)) then we will replace \( (1 + \tau)^{-\gamma(p)} \) (respectively, \( (1 + \tau)^{-\gamma(q)} \)) by \( (\log(e + t))^{-1} \).

In the following we denote by \( E_{1}(t, x) \) and \( E_{1}(t, x) \) the fundamental solutions to the linear equation, corresponding to the two initial data, namely the solution for the linear problem (1.11) with the Cauchy data \( (u_{0}, u_{1}) \) is given by

\[
u = E_{0}(t, x) * x u_{0}(x) + E_{1}(t, x) * x u_{1}(x).
\]

We define the integral operator \( N : (u, v) \in X(t) \to N[u, v] \in X(t) \) by:

\[
N[u, v] = L + (Fv, Gu),
\]

where

\[
L = E_{1}(t, x) * x (u_{0}, v_{0})(x) + E_{1}(t, x) * x (u_{1}, v_{1})(x),
\]

\[
F(v) = \int_{0}^{t} E_{1}(t - \tau, x) * x |D^{\sigma} v(\tau, x)|^{p} d\tau,
\]

\[
G(u) = \int_{0}^{t} E_{1}(t - \tau, x) * x |D^{\sigma} u(\tau, x)|^{q} d\tau.
\]

The local and global existence of small data solutions to (1.4) will follow by the standard contraction argument if we can show that for the exponents \( p, q \) satisfying the given conditions the estimates

\[
\| N[u, v] \|_{X(t)} \leq \mathcal{A} + \| (u, v) \|_{X(t)}^{p} + \| (u, v) \|_{X(t)}^{q},
\]

and the Lipschitz property

\[
\| N[u, v] - N[\tilde{u}, \tilde{v}] \|_{X(t)} \leq \| (u, v) \|_{X(t)}^{p-1} + \| (\tilde{u}, \tilde{v}) \|_{X(t)}^{p-1} + \| (u, v) \|_{X(t)}^{q-1} + \| (\tilde{u}, \tilde{v}) \|_{X(t)}^{q-1}
\]

must hold.
First, we will prove the inequality (3.20).

By the linear estimates (1.12)-(1.14) in Prop. 1.1, it immediately follows that

$$\|L\|_{X(t)} \lesssim A.$$ 

Now we will estimate the $L^2$ and $\hat{H}^\sigma$ norms of $Fv$ and $Gu$.

We use the $L^1 \cap L^2 - L^2$ estimates if $\tau \in [0, t/2]$ and the $L^2 - L^2$ estimates if $\tau \in [t/2, t]$. Then

$$\|\partial_t^j |D|^{\sigma} Fv\|_{L^2} \lesssim \int_0^{t/2} (1 + t - \tau)^{- \frac{n - (k/2 + j)}{2\sigma}} \|D^{\alpha} v(\tau, \cdot)\|_{L^1 \cap L^2} d\tau$$

$$+ \int_{t/2}^t (1 + t - \tau)^{1-(3k/2+j)} \|D^{\alpha} v(\tau, \cdot)\|_{L^2} d\tau,$$

where $j, k = 0, 1$ and $(j, k) \neq (1, 1)$. We will estimate $\|D^{\alpha} v(\tau, \cdot)\|_{L^2}$ in $L^1 \cap L^2$ and $L^2$.

Obviously $\|D^{\alpha} v(\tau, \cdot)\|_{L^1 \cap L^2} \lesssim \|D^{\alpha} v(\tau, \cdot)\|_{L^2} + \|D^{\alpha} v\|_{L^{2p}},$ and $\|D^{\alpha} v\|_{L^2} = \|D^{\alpha} v\|_{L^{2p}}$.

We apply the fractional Gagliardo-Nirenberg inequality (see [5], [2] and [7] and the Appendix for the formulation, proof and notations) with the interpolation exponents $\theta, \sigma(p, 2)$ and $\theta, \sigma(2p, 2)$ from the interval $[0, 1)$. This gives the condition $2 \leq p < \frac{n}{n + 2(\alpha - \sigma)}$. Accordingly:

$$\|D^{\alpha} v(\tau, \cdot)\|_{L^1 \cap L^2} \lesssim (1 + \tau)^{- \frac{p(n + \alpha) + n}{2\sigma} + \gamma(q)} \|(u, v)\|_{X(t)}^{p},$$

because of $\theta, \sigma(p, 2) < \theta, \sigma(2p, 2)$, meanwhile

$$\|v(\tau, \cdot)\|_{L^2} \lesssim (1 + \tau)^{p(\alpha - \theta, \sigma(2p, 2)) + \gamma(q)} \|(u, v)\|_{X(t)}^{p},$$

Combining the last estimates we conclude

$$\|\partial_t^j |D|^{\sigma} Fv\|_{L^2} \lesssim (1 + t)^{- \frac{n - (k/2 + j)}{2\sigma}} \|(u, v)\|_{X(t)}^{p} \int_0^{t/2} (1 + \tau)^{- \frac{p(n + \alpha) + n}{2\sigma} + \gamma(q)} d\tau$$

$$+ (1 + t)^{- \frac{p(n + \alpha) - n/2}{2\sigma}} \|(u, v)\|_{X(t)}^{p} \int_{t/2}^t (1 + t - \tau)^{1-(3k/2+j) + \gamma(p)} d\tau,$$

and similarly for $Gu$:

$$\|\partial_t^j |D|^{\sigma} Gu\|_{L^2} \lesssim (1 + t)^{- \frac{n - (k/2 + j)}{2\sigma}} \|(u, v)\|_{X(t)}^{p} \int_0^{t/2} (1 + \tau)^{- \frac{q(n + \alpha) + n}{2\sigma} + \gamma(p)} d\tau$$

$$+ (1 + t)^{- \frac{q(n + \alpha) - n/2}{2\sigma}} \|(u, v)\|_{X(t)}^{p} \int_{t/2}^t (1 + t - \tau)^{1-(3k/2+j) + \gamma(p)} d\tau.$$
If $p,q > \frac{n+2\sigma}{n+\alpha} = p_E$, then $\gamma(p) = \gamma(q) = 0$, and the terms $(1+\tau)^{-\frac{p(n+\alpha)+n}{2\sigma}}$, $(1+\tau)^{-\frac{q(n+\alpha)+n}{2\sigma}}$ are integrable.

Moreover, we have

$$
(1+t)^{-\frac{p(n+\alpha)-n/2}{2\sigma}} \| (u,v) \|_{X(t)}^p \int_{t/2}^t (1+t-\tau)^{-(3k/2+j)} d\tau
$$

$$
= (1+t)^{-\frac{p(n+\alpha)-n/2}{2\sigma}} \| (u,v) \|_{X(t)}^p \int_0^{t/2} (1+\tau)^{-(3k/2+j)} d\tau \lesssim (1+t)^{-\frac{q}{2\sigma}-(k/2+j)} \| (u,v) \|_{X(t)}^q.
$$

Now consider the more interesting case when $\min\{p,q\} \leq p_E$. Suppose that $q \geq p$. Then our condition implies $q > p_E$ and $\gamma(q) = 0$. In that case

$$
\int_0^{t/2} (1+\tau)^{-\frac{p(n+\alpha)+n}{2\sigma}} d\tau \lesssim \begin{cases} 
(1+t)^{\gamma(p)} \text{ if } p < p_E, \\
\log(e+t) \text{ if } p = p_E.
\end{cases}
$$

On the other hand, it is easy to verify that condition (1.5) is equivalent to

$$
-\frac{q(n+\alpha)+n}{2\sigma} + q\gamma(p) < -1,
$$

therefore the term $(1+\tau)^{-\frac{q(n+\alpha)+n}{2\sigma}+q\gamma(p)}$ again is integrable over $(0,t/2)$.

The second integrals over $(t/2,t)$ are easier to be estimated thanks to the condition $p,q \geq p_E - 1$.

Summarizing we will arrive at the estimates of the $L^2$ and $\dot{H}^s$ of $Fv$ and $Gu$. The $L^2$ norms of $(Fv)_t$ and $(Gu)_t$ can be estimated in the similar way if we again apply the linear estimates from Prop. 1.1.

Thus the first inequality (3.20) has been verified. The second inequality (3.21) can be proved by the same arguments as in the proof of Theorem 2.1, with the application of Kato - Ponce’s inequality for the homogeneous Sobolev spaces.

Theorem 2.2 thus has been proved completely.

\textbf{Remark 3.1.} It is obvious, thanks to the linear estimates, that the statement of Theorem 2.1 remains valid for the following model

$$
\begin{cases}
&u_{tt} + a_1(x)(-\Delta)^s u + b_1 u_t = |v_t|^p, \ t \geq 0, x \in \mathbb{R}^n, \\
&v_{tt} + a_2(x)(-\Delta)^s v + b_2 v_t = |u_t|^q, \ t \geq 0, x \in \mathbb{R}^n, \\
&(u,u_t,v,v_t)(0,x) = (u_0,u_1,v_0,v_1)(x), \ x \in \mathbb{R}^n,
\end{cases}
$$

where $a_1(x), a_2(x)$ are continuous functions, $b_1, b_2$ are positive constants, and

$$
0 < C_1 < a_1(x) < C_2,
$$

for $i = 1, 2$ with some positive constants $C_1, C_2$.

4. \textbf{Appendix}

4.1. \textbf{The fractional Gagliardo - Nirenberg inequality.} We will present here the formulation of the fractional Gagliardo - Nirenberg inequality in homogeneous Sobolev spaces (see [5]) that follows from the more generalized Gagliardo - Nirenberg inequalities for the Besov spaces.
Proposition 4.1 (Fractional Gagliardo - Nirenberg inequality for the homogeneous Sobolev spaces). Let \( a \in (0, \sigma) \). Then for all \( m \in (1, \infty) \):

\[
\|D^\nu u\|_{L^p} \lesssim \|D^\sigma u\|_{L^m_{\mu}}^\theta_{\alpha,\sigma}(q,m) \|u(\tau, \cdot)\|_{L^m_{\mu}}^{1-\theta_{\alpha,\sigma}(q,m)}
\]

for all \( u \in \dot{H}^{\sigma,m} \), whenever the condition

\[
\theta_{\alpha,\sigma}(q,m) := \frac{n}{\sigma} \left( \frac{1}{m} - \frac{1}{q} + \frac{1}{n} \right) \in \left[ \frac{a}{\sigma}, 1 \right).
\]

is satisfied

We note that the condition \( \theta_{\alpha,\sigma}(q,m) \in \left[ \frac{a}{\sigma}, 1 \right) \) is equivalent to

\[
m \leq q < \frac{mn}{n + m(a - \sigma)}.
\]

4.2. Composition result. We introduce the class \( \text{Lip} \mu \) in the following (see [12]).

Definition 4.2. Let \( \mu > 0, N \in \mathbb{N}_0 \) and \( 0 < \alpha \leq 1 \) such that \( \mu = N + \alpha \). Then we define

\[
\text{Lip} \mu = \left\{ f \in C^{N, \text{loc}}(\mathbb{R}) : f^{(j)}(0) = 0, j = 0, \ldots, N, \text{ and } \sup_{t_0 \neq t_1} \frac{|f^{(N)}(t_0) - f^{(N)}(t_1)|}{|t_0 - t_1|^\alpha} < \infty \right\}.
\]

Further we put

\[
\|f\|_{\text{Lip} \mu} = \sum_{j=0}^{N-1} \frac{|f^{(j)}(t)|}{|t|^{\mu-j}} + \sup_{t_0 \neq t_1} \frac{|f^{(N)}(t_0) - f^{(N)}(t_1)|}{|t_0 - t_1|^\alpha}.
\]

It is clear that \( |t|^\mu \in \text{Lip} \mu \), \( t|t|^{\mu-2} \in \text{Lip} (\mu - 1) \) for \( \mu > 1 \). The following helpful general composition result for the class \( \text{Lip} \mu \) was obtained in [12]. Let us denote \( \sigma_p = n \max \{ 0, \frac{1}{p} - 1 \} \).

Proposition 4.3 (Theorem 6.3.4 (i) in [12]). Let \( \sigma_p < s < \mu \) and \( \mu > 1 \).

Then there exists some constant \( c \) such that

\[
\|G(f)\|_{F_{p,q}^\mu} \leq c \|G\|_{L^p \mu} \|f\|_{L^p_{\mu}} \|f\|_{L^\infty}^{\mu-1}
\]

holds for all \( f \in F_{p,q}^\mu \cap L^\infty \) and all \( G \in \text{Lip} \mu \).

Proposition 4.3 together with the Sobolev embedding imply immediately the following consequence in the supercritical case \( s > \frac{n}{2} \).

Corollary 4.4 (Composition result). Let \( s \in (\frac{n}{2}, p) \). Denote either \( G(u) = |u|^p \) or \( G = \pm u|u|^{p-1} \) with \( p > 1 \). Then for all \( u \in H^s \) the following composition estimate holds:

\[
\|G(u)\|_{H^s} \lesssim \|u\|_{H^s}^p.
\]

In the homogeneous spaces, we can obtain the composition result by the following estimate

Proposition 4.5. Let \( p > 1, 1 < r < \infty \) and \( u \in H^{s,r} \), where \( s \in (\frac{n}{2}, p) \). Let us denote by \( F(u) \) one of the functions \( |u|^p, \pm u|u|^{p-1} u \) with \( p > 1 \). Then, the following estimate holds:

\[
\|F(u)\|_{H^{s,r}} \lesssim \|u\|_{H^{s,r}} \|u\|_{L^\infty}^{p-1}.
\]
Corollary 4.6. Under the assumptions of Proposition 4.5 it holds
\[ \|F(u)\|_{\dot{H}^{s,r}} \lesssim \|u\|_{\dot{H}^{s,r}} \|u\|_{L^\infty}^{p-1}. \]
The proof can be found in [7].

Proposition 4.7 (The Kato-Ponce inequality for homogeneous Sobolev spaces). For all functions \( f \in \dot{H}^{s,p_2} \cap L^{q_1} \) and \( g \in \dot{H}^{s,q_2} \cap L^{p_1} \) it holds
\[ \|D^s(fg)\|_{L^r} \lesssim \|f\|_{L^{p_1}} \|D^s g\|_{L^{q_1}} + \|D^s f\|_{L^{p_2}} \|g\|_{L^{q_2}}, \]
where \( s > 0 \) and \( \frac{1}{r} = \frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{p_2} + \frac{1}{q_2} \) for \( 1 < r < \infty, \ 1 < p_1, q_2 \leq \infty, 1 < p_2, q_1 < \infty. \)
The proof of this harmonic analysis result can be found in [4].

References
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